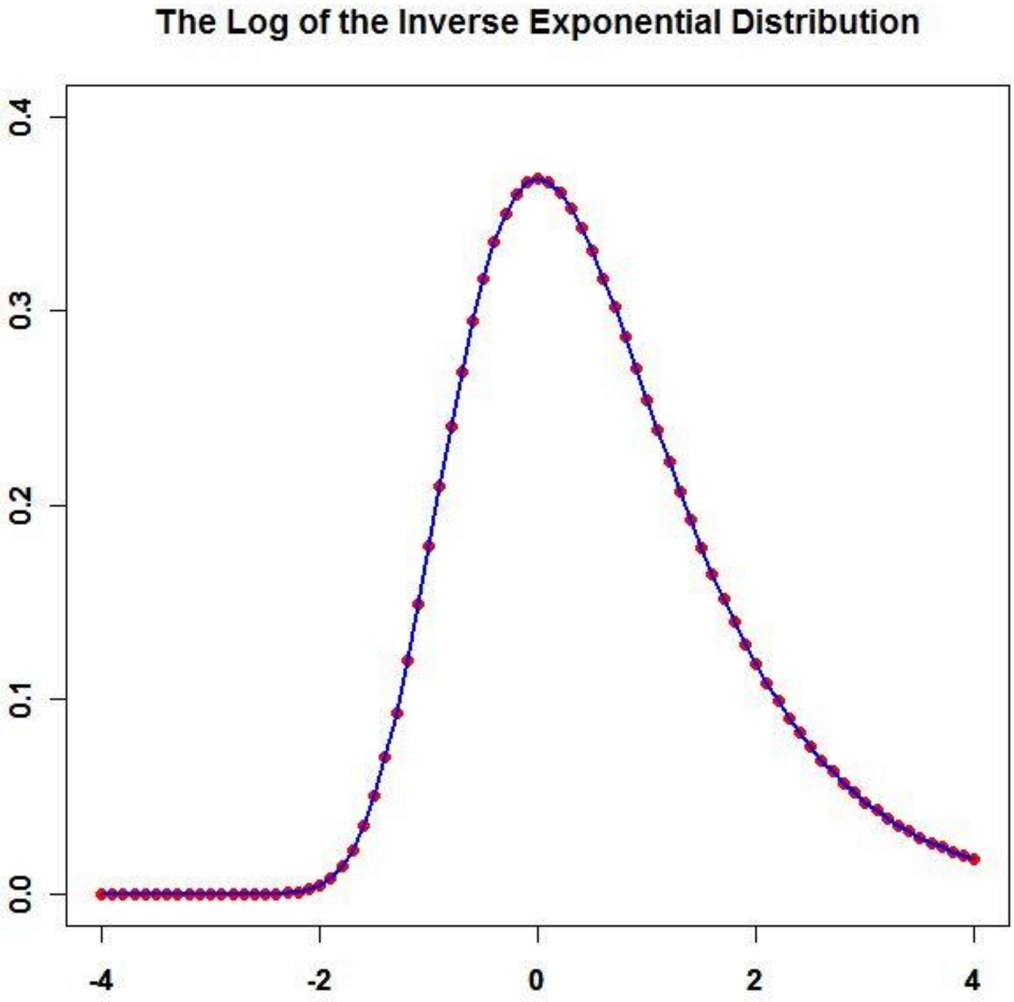
Derivation of the Logit Probability

Utility function for Yea and Nay choices:

$$\mathbf{U_{iy} = e^{-d_{iy}^2} + \varepsilon_{iy} \quad and \quad U_{in} = e^{-d_{in}^2} + \varepsilon_{in}} \qquad \textbf{(1)}$$

where $\mathbf{d_{iy}^2}$ and $\mathbf{d_{in}^2}$ are the squared distances from the ith legislator to the Yea and Nay choices and the $\boldsymbol{\varepsilon}$ are distributed as the logarithm of the inverse of an exponential variable (Dhrymes, 1978, p. 342). Namely

$$\mathbf{f(\varepsilon) = e^{-\varepsilon}e^{-e^{-\varepsilon}},\ -\infty < \varepsilon < +\infty} \qquad \textbf{(2)}$$



The probability that the legislator will choose the Yea alternative is:

$$\mathbf{P(U_{iy} > U_{in}) = P(e^{-d_{iy}^2} + \varepsilon_{iy} > e^{-d_{in}^2} + \varepsilon_{in}) = P(e^{-d_{iy}^2} - e^{-d_{in}^2} > \varepsilon_{in} - \varepsilon_{iy}) =}$$

$$\mathbf{P(\varepsilon_{in} - \varepsilon_{iy} < e^{-d_{iy}^2} - e^{-d_{in}^2}) = P(\varepsilon_{iy} - \varepsilon_{in} > e^{-d_{in}^2} - e^{-d_{iy}^2})} \quad \textbf{(3)}$$

In order to get the distribution of $\varepsilon_{iy} - \varepsilon_{in}$ set up the joint density and then do a change of variables (note that the distribution of $\varepsilon_{in} - \varepsilon_{iy}$ will be the same as the distribution of $\varepsilon_{iy} - \varepsilon_{in}$):

$$\mathbf{f(\varepsilon_{iy}, \varepsilon_{in}) = e^{-(\varepsilon_{iy} + \varepsilon_{in})} e^{-(e^{-\varepsilon_{iy}} + e^{-\varepsilon_{in}})}} \qquad \textbf{(4)}$$

Set $\mathbf{y = \varepsilon_{iy} - \varepsilon_{in}}$ **and** $\mathbf{z = \varepsilon_{in}}$

Hence $\mathbf{\varepsilon_{iy} = y + z}$ **and** $\mathbf{\varepsilon_{in} = z}$

and the Jacobian is:

$$\mathbf{J = \begin{vmatrix} \frac{\partial \varepsilon_{iy}}{\partial y} & \frac{\partial \varepsilon_{iy}}{\partial z} \\ \frac{\partial \varepsilon_{in}}{\partial y} & \frac{\partial \varepsilon_{in}}{\partial z} \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} = 1}$$

Hence

$$\mathbf{f(y+z, z) = e^{-(y + 2z)} e^{-(e^{-(y+z)} + e^{-z})} = e^{-y} e^{-2z} e^{-[e^{-z}(1+e^{-y})]}}$$

To get the distribution of $\mathbf{y = \varepsilon_{iy} - \varepsilon_{in}}$ integrate out **z**:

$$\mathbf{\int_{-\infty}^{+\infty} e^{-y} e^{-2z} e^{-[e^{-z}(1+e^{-y})]} dz} \qquad \textbf{(5)}$$

This requires another change in variables:

Set $\mathbf{v = e^{-z}(1+e^{-y})}$

Note that $\mathbf{0 < v < \infty}$ **because** $\mathbf{0 < e^{-z} < \infty}$ **as** $\mathbf{-\infty < z < \infty}$

Hence, $\ln(v) = -z + \ln(1+e^{-y})$

and $z = \ln(1+e^{-y}) - \ln(v)$

Therefore, $\frac{\partial z}{\partial v} = -\frac{1}{v}$ **and** $J = \left|\frac{\partial z}{\partial v}\right| = \frac{1}{v}$

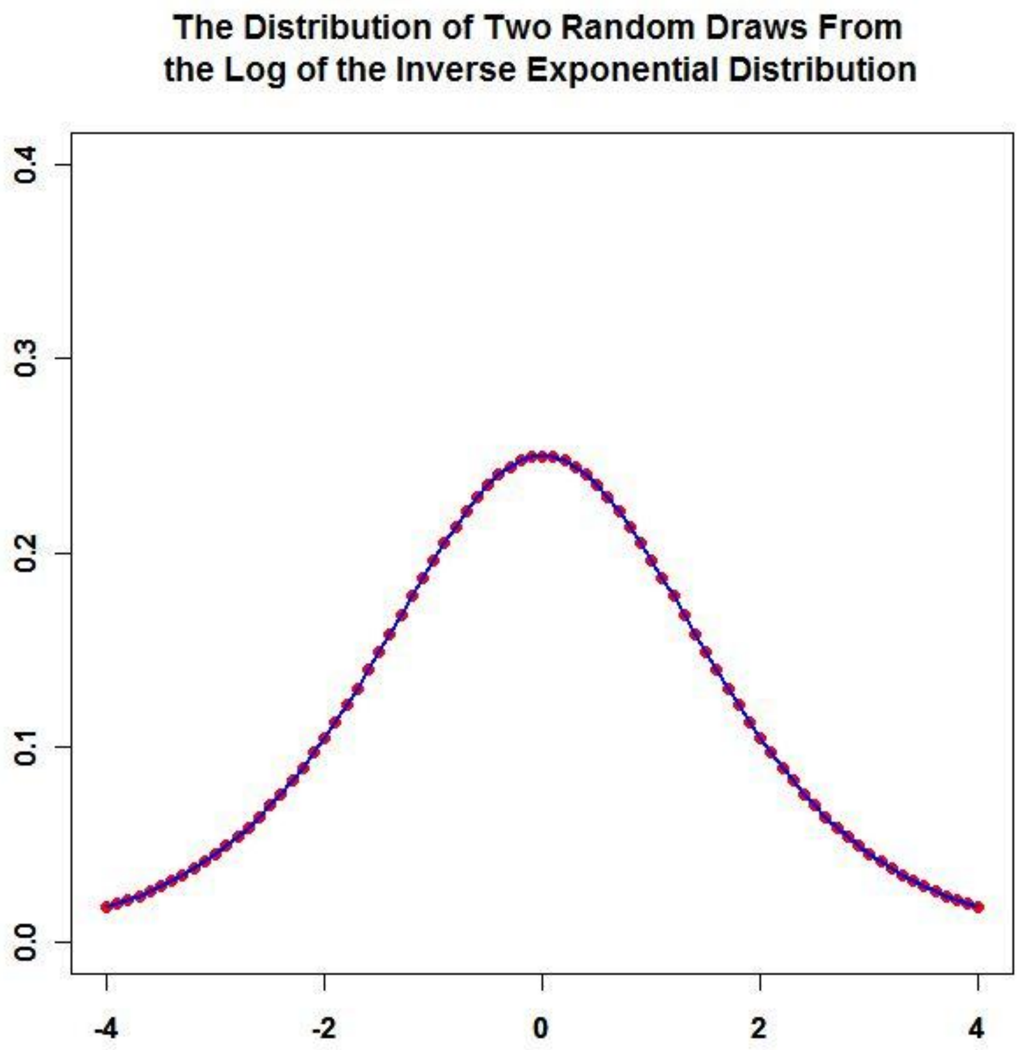
Hence

$$\int_{-\infty}^{+\infty} e^{-y} e^{-2z} e^{-[e^{-z}(1+e^{-y})]} dz = e^{-y} \int_{0}^{+\infty} e^{-2\ln(1+e^{-y})} e^{2\ln(v)} e^{-\{e^{-[\ln(1+e^{-y})-\ln(v)]}[1+e^{-y}]\}} \frac{1}{v} dv =$$

$$e^{-y} \int_{0}^{+\infty} (1+e^{-y})^{-2} v^2 e^{-\{(1+e^{-y})^{-1} v[1+e^{-y}]\}} \frac{1}{v} dv = e^{-y}(1+e^{-y})^{-2} \int_{0}^{+\infty} v^2 e^{-v} \frac{1}{v} dv =$$

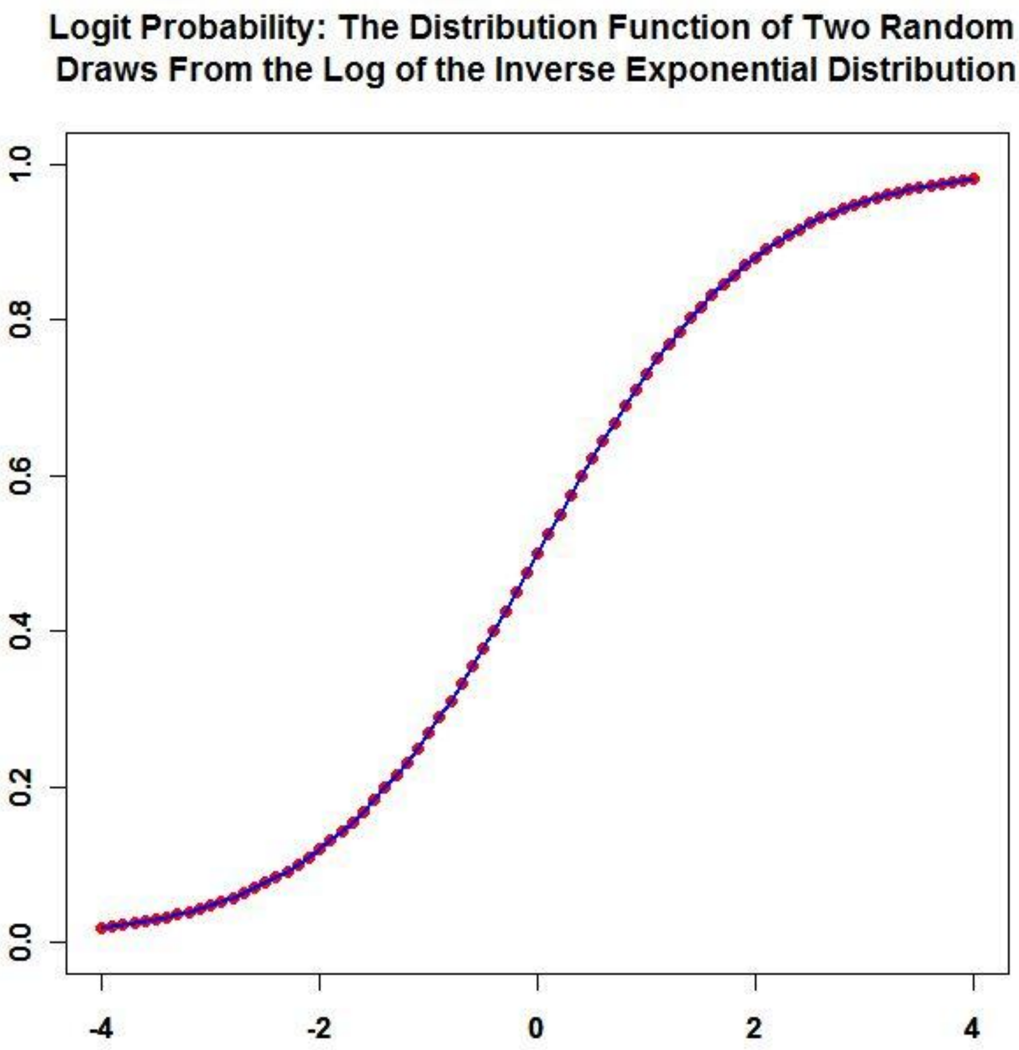
$$e^{-y}(1+e^{-y})^{-2} \int_{0}^{+\infty} v e^{-v} dv = e^{-y}(1+e^{-y})^{-2}\Gamma(2) = e^{-y}(1+e^{-y})^{-2} \qquad (6)$$

Therefore, $f(\varepsilon_{iy} - \varepsilon_{in}) = f(y) = \frac{e^{-y}}{(1+e^{-y})^2}$, **(7)**



To get the distribution function:

$$\mathbf{F(y<t) = \int_{-\infty}^{t} \frac{e^{-y}}{(1+e^{-y})^2}dy = \Big._{-\infty}^{t}\Big| \frac{1}{1+e^{-y}} = \frac{1}{1+e^{-t}}}$$



Hence

$$\mathbf{P(\varepsilon_{in} - \varepsilon_{iy} < e^{-d_{iy}^2} - e^{-d_{in}^2}) = \frac{1}{1 + e^{-(e^{-d_{iy}^2} - e^{-d_{in}^2})}}} \quad \textbf{(8)}$$